$n = 2$ (Simpson's $\frac{1}{3}$ Rule) :-

Here no. of node is 3 . which are

$$x_0 = a \;,\; x_1 = \frac{a+b}{2} \;,\; x_2 = b$$

so $I(f) = R_3(f) = w_0 f(a) + w_1 f\left(\frac{a+b}{2}\right) + w_2 f(b)$

$$w_0 = \int_a^b l_0(x)\, dx = \int_a^b \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}\, dx$$

$$= \int_0^2 \frac{(x-1)h\,(x-2)h}{-h\,(-2h)}\, h\, dx \qquad \left[\begin{array}{l} x = x_0 + xh \\ x_i = x_0 + ih \end{array}\right.$$

$$= \frac{h}{2} \int_0^2 (\alpha-1)(\alpha-2)\, d\alpha$$

$$= \frac{h}{2} \int_0^2 (\alpha^2 - 3\alpha + 2)\, d\alpha$$

$$= \frac{h}{2} \left[\frac{\alpha^3}{3} - \frac{3\alpha^2}{2} + 2\alpha \right]_0^2$$

$$= \frac{h}{2} \left(\frac{8}{3} - 6 + 4 \right)$$

$$= \frac{h}{2} \times \frac{2}{3} = \frac{h}{3}$$

$$= \frac{b-a}{6}$$

$\left[\because h = \frac{b-a}{n} \text{ and here } n = 2 \right]$

$$w_1 = \int_a^b l_1(x)\, dx$$

$$= \int_a^b \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}\, dx$$

$$= \int_0^2 \frac{\alpha h\, (\alpha-2)h}{h(-h)}\, h\, d\alpha$$

$$= -h \int_0^2 \alpha(\alpha-2)\, d\alpha$$

$$= (-h) \left[\frac{\alpha^3}{3} - 2\frac{\alpha^2}{2} \right]_0^2$$

$$= -h \left(\frac{8}{3} - 4 \right) = -h\left(\frac{-4}{3}\right)$$

$$= \frac{4(b-a)}{6}$$

$$w_2 = \int_a^b l_2(x)\, dx$$

$$= \int_a^b \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}\, dx$$

$$= \int_0^2 \frac{\alpha h\, (\alpha-1)h}{2h \cdot h}\, h\, d\alpha$$

$$= \frac{h}{2} \int_0^2 \alpha(\alpha-1)\, d\alpha$$

$$= \frac{h}{2} \left[\frac{\alpha^3}{3} - \frac{\alpha^2}{2} \right]_0^2$$

$$= \frac{h}{2} \left(\frac{8}{3} - \frac{4}{2} \right)$$

$$= \frac{h}{2} \cdot \frac{2}{3} = \frac{h}{3}$$

$$= \frac{b-a}{6}$$

So $I(f) = \frac{b-a}{6} f(x_0) + 4\frac{(b-a)}{6} f(x_1) + \frac{b-a}{6} f(x_2)$

$$\Rightarrow \boxed{I(f) = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right]} \quad \text{—(5)}$$

is called Simpson's $\frac{1}{3}$ rule.
This is also called parabolic rule.

Theorem

[For a quadrature rule based on n+1 nodes, the number n is defined to be the index of the rule]

(a) If n is even. If $f \in C^{n+2}[a,b]$

then $E(f) = \frac{h^{n+3} f^{(n+2)}(\eta_1)}{(n+2)!} \int_0^n \mu^2(\mu-1)\cdots(\mu-n)\, d\mu$

(b) If n is odd, $f \in C^{n+1}[a,b]$

then $E(f) = \frac{h^{n+2} f^{(n+1)}(\eta_2)}{(n+1)!} \int_0^n \mu(\mu-1)\cdots(\mu-n)\, d\mu$

where $\eta_1, \eta_2 \in [a,b]$ & $h = \frac{b-a}{n}$.

Error in Simpson's $\frac{1}{3}$ rule :-

As here the index of Simpson's $\frac{1}{3}$ rule is 2 (even) so the Error is given by.

$$E(f) = \frac{h^5}{4!} f^{iv}(\eta) \int_0^2 y^2(y-1)\cdot(y-2)\,dy$$

where $h = \frac{b-a}{2}$

But $\int_0^2 y^2(y-1)(y-2)\,dy$

$$= \int_0^2 (y^4 - 2y^3 - y^3 + 2y^2)\,dy$$

$$= \frac{y^5}{5} - \frac{3y^4}{4} + \frac{2y^3}{3}\Big]_0^2$$

$$= \frac{33}{5} - 12 + \frac{16}{3} = \frac{96 - 180 + 80}{15}$$

$$= \frac{-4}{15}$$

So $E(f) = \frac{h^5}{24} \cdot f^{iv}(\eta) \frac{(-4)}{15}$

$$\boxed{E(f) = -\frac{h^5}{90} f^{iv}(\eta)}, \quad a \le \eta \le b \qquad \text{---------- (6)}$$

$$\Rightarrow E(f) = \frac{(b-a)^5}{2880} f^{iv}(\eta)$$

$n = 3$ (Simpson's $\frac{3}{8}$ rule)

Here no. of node is 4., these are $x_0 = a$, $x_1 = x_0 + h$, $x_2 = x_0 + 2h$, $x_3 = b$., where $h = \frac{b-a}{3}$.

$$I(f) = R_4(f)$$

$$= w_0 f_0 + w_1 f_1 + w_2 f_2 + w_3 f_3$$

$$w_0 = \int_a^b l_0(x)\,dx = \int_a^b \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}\,dx$$

$$= \int_0^3 \frac{(\alpha-1)h\,(\alpha-2)h\,(\alpha-3)h}{-h\,(-2h)\,(-3h)}\,h\,d\alpha$$

$$= -\frac{h}{6}\int_0^3 (\alpha-1)(\alpha-2)(\alpha-3)\,d\alpha$$

$$= -\frac{h}{6_2} \times \left(-\frac{9}{4}\right) = \frac{3h}{8}$$

Similarly,

$$w_1 = \frac{9h}{8}, \quad w_2 = \frac{9h}{8}, \quad w_3 = \frac{3h}{8}$$

$$I(f) = \frac{3h}{8} f(x_0) + \frac{9h}{8} f(x_1) + \frac{9h}{8} f(x_2) + \frac{3h}{8} f(x_3)$$

$$\Rightarrow \boxed{I(f) = \frac{3h}{8}\left[f(x_0) + 3\{f(x_1) + f(x_2)\} + f(x_3)\right]} \quad \text{---(7)}$$

which is called as Simpson's $\frac{3}{8}$ rule.

Error in Simpson's $\frac{3}{8}$ rule :-

As the index (n) of the rule is odd.

$$\text{So } E(f) = \frac{h^5}{4!} f^{iv}(\eta) \int_0^3 \mu(\mu-1)(\mu-2)(\mu-3)\,d\mu$$

$$\text{But } \int_0^3 \mu(\mu-1)(\mu-2)(\mu-3)\,d\mu = -\frac{9}{10}$$

$$\text{So } E(f) = \frac{h^5}{8\,24} f^{iv}(\eta)\left(-\frac{9}{10}\right)$$

$$\Rightarrow \boxed{E(f) = -\frac{3h^5}{80} f^{iv}(\eta)}\,; \quad a \le \eta \le b. \quad \text{---(8)}$$